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Ques 1 a) $a_n = \left[1 + \frac{1}{n}\right]^n$

Step 1:- We will show it is monotonically increas[ing]
Applying Binomial expansion:-

$$\left[1+\frac{1}{n}\right]^n = 1 + \frac{{}^nC_1}{n} + \frac{{}^nC_2}{n^2} + \cdots + \frac{{}^nC_{n-1}}{n^{n-1}} + \frac{{}^nC_n}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left[1-\frac{1}{n}\right] + \frac{1}{3!}\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right] + \cdots$$

$$= \sum_{k=0}^{n} \frac{1}{k!}\left[1-\frac{1}{n}\right] \cdots \times \left[1-\frac{k-1}{n}\right]$$

As $1-\frac{k-1}{n} < 1-\frac{k-1}{n+1}$

$\Rightarrow$ $a_n < a_{n+1}$

Hence $\{a_n\}$ monotonically increasing.

(2 marks)

Step 2:- We will show it is Bounded above

Since $\frac{1}{k!} < \frac{1}{2^{k-1}}$, we have

$$a_n < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right)$$

$$< 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} = 3$$

(2 marks)

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b) $a_n = \dfrac{n^n}{n!}$

Using ratio test for sequences, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{[n+1]^{n+1}}{[n+1]!} \times \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \left[\frac{n+1}{n}\right]^n$$

$$= e$$

(2marks)

As $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = e > 1$

This implies the sequence <u>diverges</u>

(1 mark)

For Q2a)

For those using direct definition use own judgement

Mention $\sum_{k=1}^{\infty} \frac{1}{k!} < \infty$ 1

Seq. of partial sums conv. 1

conv. $\Rightarrow$ cauchy 1

$|a_{n+1} - a_n| \leq \frac{1}{2} |a_n - a_{n-1}|$ 2.5

Mention of contraction result to conclude 0.5

Monotonic 0.5

bounded 1

bounded + monotonic $\Rightarrow$ conv. 0.5

conv. $\Rightarrow$ Cauchy 1

For Q2.b)

Showed monotonicity 0.5

Showed bounded 1.5

bounded + monotone $\Rightarrow$ conv. 0.5

conv. $\Rightarrow$ cauchy 0.5

$|a_{n+1}| < 3$ 1.5

$|a_{n+1} - a_n| < 3/2^n$ 0.5

Using this showed Cauchy 0.5

3) a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{\left(-\frac{n}{n+1} - (-1)^n\right)}}{n}$

Sol1: $\displaystyle = \sum_{k=1}^{\infty} \frac{3^{\left(\frac{-2k}{2k+1} - (-1)^{2k}\right)}}{2k} + \sum_{k=0}^{\infty} \frac{3^{\left(\frac{-2k+1}{2k+2} - (-1)^{2k+1}\right)}}{2k+1}.$

$$= \sum_{k=1}^{\infty} \frac{3^{\left(\frac{-2k}{2k+1} - 1\right)}}{2k} + \sum_{k=0}^{\infty} \frac{3^{\left(\frac{-2k+1}{2k+2} + 1\right)}}{2k+1}$$

$$= \sum_{k=1}^{\infty} \frac{3^{\left(\frac{-2k-2k-1}{2k+1}\right)}}{2k} + \sum_{k=0}^{\infty} \frac{3^{\left(\frac{-2k-1+2k+2}{2k+2}\right)}}{2k+1}.$$

$$= \sum_{k=1}^{\infty} \frac{3^{\left(\frac{-4k-1}{2k+1}\right)}}{2k} + \sum_{k=0}^{\infty} \frac{3^{\left(\frac{1}{2k+2}\right)}}{2k+1}. \quad -1$$

$\displaystyle\sum_{k=0}^{\infty} \frac{3^{\left(\frac{1}{2k+2}\right)}}{2k+1}$ limit comparison test with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k+1}$

$$\lim_{k\to\infty} \left| \frac{\frac{3^{1/2k+2}}{2k+1}}{1/2k+1} \right| = \lim_{k\to\infty} \left| 3^{1/2k+2} \right| = 1 > 0. \quad -1$$

$\displaystyle\sum_{k=0}^{\infty} \frac{1}{2k+1}$ diverges $\Rightarrow$ $\displaystyle\sum_{k=1}^{\infty} \frac{3^{1/2k+2}}{2k+1}$ diverges. -1

$\displaystyle\sum_{k=1}^{\infty} \frac{3^{\left(\frac{-4k-1}{2k+1}\right)}}{2k}$ limit comparison with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{9(2k)}.$

$$\lim \left| \frac{\frac{3^{\frac{-4k-1}{2k+1}}}{2k}}{\frac{3^{-2}}{2k}} \right| = \lim \left| 3^{\left(\frac{-4k-1}{2k+1} + 2\right)} \right| = 1 > 0.$$

$\displaystyle\sum \frac{1}{9(2k)}$ diverges $\Rightarrow$ $\displaystyle\sum_{k=1} \frac{3^{\frac{-4k-1}{2k+1}}}{2k}$ diverges.

Sol 2:

$$\frac{n}{n+1} + (-1)^n \le \frac{3}{2} \quad - 1 \quad \text{(any number beyond } 3/2 \text{ is acceptable)}$$

$$\frac{3^{-\left(\frac{n}{n+1}\right) - (-1)^n}}{n} \geq \frac{3^{-3/2}}{n} \quad - 1 \qquad 2$$

$3^{-3/2} \sum \frac{1}{n}$ diverges $\Rightarrow$ by comparison Test $\sum a_n$ diverges. — 1

3.(b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$

$$0 < \frac{1}{n(n+1)(n+2)} < \frac{1}{n^3} \quad \forall n \geq 1$$

[1 mark]

We know that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent

[1 mark]

$\therefore$ By the comparison test, the given series converges [1 mark]

Another solution:

$$\frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

$$\Rightarrow \frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right]$$

[1 mark]

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left[\left(1 - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right)\right]$$

{1 mark}

$$\to \frac{1}{4} \text{ as } n \to \infty.$$

[1 mark]

Hence, the given series is convergent.

Test the convergence of the following series.

a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln(\ln n)}$.

Solution. We know that

$$\ln(n) \leq n$$

$$\Rightarrow \ln(\ln(n)) \leq \ln(n)$$

$$\Rightarrow \frac{1}{\ln(n)} \leq \frac{1}{\ln(\ln(n))}$$

$$\Rightarrow \frac{1}{n\ln(n)} \leq \frac{1}{n\ln(\ln(n))} . \longrightarrow 1$$

We know that $\sum \frac{1}{n\ln(n)}$ is divergent. $\longrightarrow 1$

$\therefore$ By comparison test, the given series diverges. $\longrightarrow 1$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^n}{n!}$

Solution. Let $a_n = \dfrac{2^n n^n}{n!}$. Then

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n n^n} \longrightarrow ①$$

$$= 2 \cdot \left(\frac{n+1}{n}\right)^n$$

$$= 2 \cdot \left(1 + 1/n\right)^n \longrightarrow 2e > 1. \longrightarrow ①$$

$\therefore$ By ratio test the given series diverges. $\longrightarrow ①$